CONCEPTUAL MODELLING OF FUNCTIONS BY AN EXPERIENCED TEACHER

Mohan Chinnappan
The University of Auckland
<chinnap@math.auckland.ac.nz>

Mike Thomas
The University of Auckland
<m.thomas@math.auckland.ac.nz>

An important element in teaching is the quality of content knowledge that teachers use in the design and delivery of their lessons. In this study, we present a framework for investigating how this knowledge is structured. The framework is then used in the analysis of an experienced teacher's knowledge of functions and the teaching of functions. The data show that our teacher has built up knowledge that is dominated by conceptual rather than procedural aspects of functions.

BACKGROUND

Much of the recent research on the learning of mathematics has focussed on students' ability to use previously-acquired knowledge in making progress with the solution of novel problems. An important development in this area has been appreciating that the quality of the knowledge that students acquire may have a significant influence on how well that knowledge is used in the search for solutions to problems. In a classroom setting, teachers play an active role in facilitating not only the acquisition of new knowledge by their students, but also in providing pedagogically valuable experiences that may assist in extending that knowledge into new territories.

An important element in the goals that teachers set for their lessons and the structuring of these lessons is their own understanding of both the subject-matter and their students. Thus, the nature of a teachers' knowledge base underlying a particular mathematical topic and the teaching of that topic can be expected to exert a major influence on the quality of the understanding that students develop about that area of mathematics. While this point about the role of teachers' knowledge base has received considerable support in research findings (Ball and McDiarmid, 1990) and curriculum reform documents (National Council of Teachers of Mathematics, 1989) there is little information about the interaction between teachers' subject-matter knowledge and what students learn. In the present study, we begin a process of addressing this issue by examining an experienced teacher's knowledge about algebra and how that knowledge is used in modelling functions.

Schemas as Structured Mathematical Knowledge

Network theorists have advanced several theoretical frameworks in which to investigate concepts and their development. According to one view, conceptual growth and mathematical understanding can be interpreted in terms of conceptual nodes and relations between nodes (Anderson, 1995). As students' experiences with a concept or a set of concepts increase, they come to form organised meaningful wholes. Various attempts have been made to elucidate such cognitive structures. Among these, the notion of *schemas* has gained considerable support amongst researchers.

In the context of learning, schemas have been given a number of interpretations in the psychological literature. Skemp (1985) describes how we construct 'what we already know' by engaging in mental construction of reality by building and testing a schematic knowledge structure, where a schema is "a conceptual structure existing in its own right, independently of action" (Skemp, 1979:219). In the context of problem solving, Paas (1992:429) describes how a schema "can be conceptualised as a cognitive structure that enables problem solvers to recognize problems as belonging to a particular category of problems that require particular operations to reach a solution", while Sweller (1992:47) defines a schema as "a cognitive construct that permits problem solvers to categorise problems according to the moves required to solve them." Because our existing schemas serve either to promote or restrict the association of new concepts, the quality of what an individual already knows is a key determinant of our ability to understand, or as Skemp (1979:113) concludes "our conceptual structures are a major factor of our progress".

What then do schemas comprise? Dubinsky and others (Dubinsky, 1991, Cottrill *et al.*, 1996) use the acronym APOS to describe the four components of Action, Process, Object, Schema in the building of mathematical knowledge. The chain of events, they suggest, develops as follows. Actions, when applied to objects become processes, which in turn become *encapsulated* as mental objects. In turn examples of these three link together to form cognitive structures or schemas. Thus, conceptual entities in mathematics often present themselves with two distinct but complementary faces; they may be viewed as dynamic processes or as static objects. To make a mathematical idea readily manipulable and applicable in other contexts, it must be available internally in a concise form and the encapsulation of the process as an object is one way of accomplishing this. The relations that are constructed between the conceptual objects forming a schema could represent, for example: similarities and dissimilarities between concepts; instances of a concept; procedures for using concepts for solving problems; or affective factors related to those concepts.

According to Anderson (1995), two variables determine the quality of a schema: the spread of the network and the strength of the links between the various components of information located within the network. A complex schema can be characterised as having a large network of ideas that are built around one or more core concepts. Further, the links between the various components in the network are robust, a feature which contributes to the accessing and use of the schema in problem-solving and other situations. A well structured schema can also benefit students by helping them assimilate incoming new mathematical ideas because such a schema can be expected to have many conceptual points to link with. As a theoretical construct schemas provide a useful way to interpret the growth of mathematical knowledge.

Teacher Knowledge and Schema Induction

Applying the above ideas about the nature of mathematical schemas and their formation to teaching and learning in the classroom it becomes apparent that the quality of the mental schemas of the teacher may be a key component in a) what is learnt and b) how it is learnt by their students. When we examine mental schemas of teachers in any given content area we become aware of what the important links which we want students to build into their knowledge structures are, and we could structure the learning environment towards these.

Leinhardt (1989) have suggested the existence of strong links between teachers' subject-matter knowledge, their explanations and the type of representations generated by students. A schema-based analysis, therefore, suggests that teacher actions could promote the construction of powerful schemas that would benefit student learning in two important ways. Firstly, students would better access prior knowledge and integrate that with incoming information. Secondly, students could be expected to deploy acquired knowledge flexibly during the process of problem analysis. Hence the question is, what is the nature of a teacher's knowledge that would promote the construction of type of schema that are useful?

The aim of this research study was to examine the above issue by characterising the schemas of experienced teachers in the content area of function. This is part of a larger project in which we will compare and contrast schemas of the experienced teachers with those of teachers who are new to the teaching of algebra and functions.

One major difference we have hypothesised has to do with encapsulation of processes as objects. Many individuals appear not to progress to the point where they can think in a dual *proceptual* (Gray & Tall, 1994) or *versatile* way (Tall & Thomas, 1991; Hong & Thomas, 1998), about mathematical symbols, seeing them either as a process invoked by the symbol or as the concept represented by it. Instead, they are *process-oriented* (Thomas, 1994) in their thinking, constrained primarily to mathematical processes. For a teacher the absence of the object view in their schemas may structure their thinking, causing them to over-stress procedural methods. In contrast the *versatile* teacher, with a global view of a concept, is able to see its components, or constituent processes, and relate these to the whole, rather than seeing only the part in the context of limited, procedural understanding.

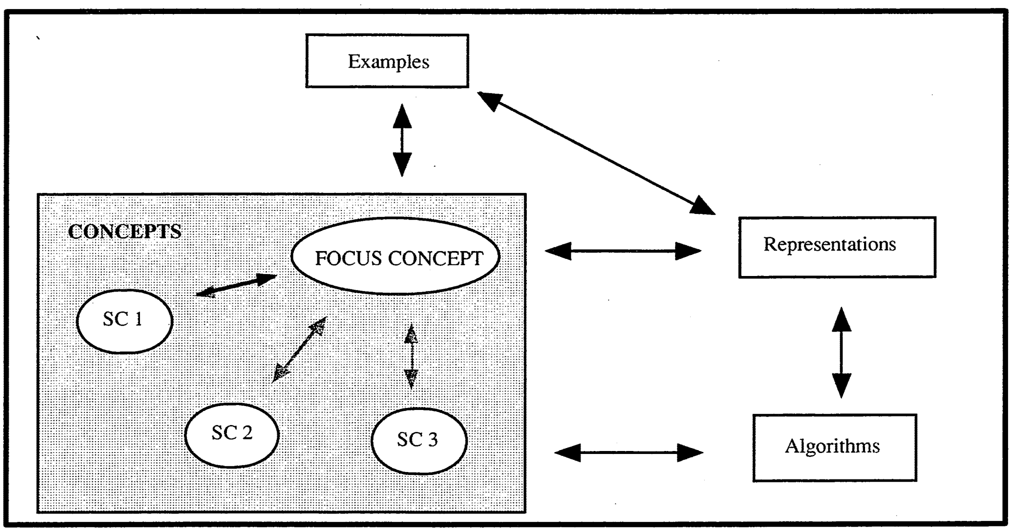
Considering this in the context of function, how would the schemas and teaching approach of a teacher with a conceptual view of function diverge from those who have a primarily procedural base? An example could be the construction of a composite function fg from the functions f and g where

$$f(x)=x^2 \text{ and } g(x)=x+1, \text{ so that } fg(x)=(x+1)^2.$$

One might be able to carry out such a procedure without actually having a concept of what a composite function actually is. Our view is that it would be difficult for a procedurally-oriented teacher to engage students from a conceptual perspective. What does this kind of teaching involve?

We present in Figure 1 a theoretical view of a modelling approach to teaching. According to this model, the emphasis is on a progression from the *formal aspects* to the *algorithmic components* associated with the focus concept (Fischbein, 1994). It should be noted that this process is usually cyclical. This model may be contrasted with a procedural perspective which starts with a symbolic representation for a function, seen as representing a procedure and either operates on it (as above) or uses it as an algorithm in the form of value in and value out.

Figure 1
A macro model of the structure of teachers' mathematical knowledge



Note: SC 1 stands for subsidiary concept one etc.

METHOD AND RESULTS

This research employs a case study methodology, examining the conceptual structures of individual teachers and their influence on their teaching. The research is very much still in progress and this paper describes the results in relation to one participant experienced teacher.

Margot[1], an Experienced Teacher

A number of criteria were set up for defining whether a teacher could be categorised as experienced, particularly with respect to the use of technology in their teaching, which is to be a future focus of the study. Margot, a secondary school, teacher in Auckland, New Zealand, fulfilled each of these, having 31 years teaching experience, including 15 years using technology in mathematics teaching. She has been active in promoting the use of calculators and computers to other teachers in her school, has attended professional development courses including study for a Diploma in Mathematics Education, where she took a paper on Technology in Mathematics Education. In addition, she has run advisory courses on using graphic calculators and was recently seconded to a one-year appointment as a mathematics adviser.

Procedure

Margot took part in a non–structured, free recall interview in January 1999, where she was asked to talk freely about functions and polynomials. The interview was recorded on audiotape and afterwards transcribed for analysis. Later we were able to go into Margot's school and observe and videotape two lessons on function. This data is still being analysed and the discussion presented here is based solely on the free recall interview.

Margot's Schematic Structure

Margot had a conceptual view of function underpinning her teaching. She saw function very much as a relationship between two variables and commented that, when teaching about functions "you tend to concentrate on is the relationship between two variables the fact that there is one variable affecting the outcome of another variable. . . So you're encouraging relationships between variables." Further this relationship was about change. She commented that "You could just say that there is a connection between these two variables, which one is causing the change, and what is the result of the change?" and "So what we tend to do is do this practical type work first where they're getting the idea of this variable changing and this one resulting." This relationship for her was strongly based on the idea of a one-to-one mapping between the values of the two variables. Her comments were often expressed in terms of practical examples, which she clearly saw as very important for her own understanding and her teaching, and as modelling the function concept. For example she talks about beakers of water, kangaroo jumps, pendulums and costs etc.

Margot emphasises throughout the one–to–one nature of a function, "one in, one out" clearly excluding a one–to–many relationship (this emphasis should not necessarily be taken to mean that she excludes a many–to–one function, since this was not specifically mentioned). Furthermore, the distinction between discrete and continuous variables is something she is very conscious of.

> One of the things that I find that causes confusion is the distinction between discrete and continuous. You take them away and you do, you know a quadratic patterning, your kangaroos jumping or whatever and that's a discrete pattern. And then all of a sudden you produce a parabola which is continuous, and I don't think myself at the moment that I'm yet very good at making the distinction there for them between the two, and I think a lot of them lose that.. .Yes, yes, because the kangaroos they can see, where the kangaroo, the jumping one, they can see that this is a discrete, you know, you take ten jumps, you take five jumps you take four jumps, whereas with this they will be able to see that with that fact that what you're graphing is a continuous, a continuous movement.

What subsidiary concepts did Margot have embedded in her overall conceptual view of function? Clearly from her comments variable is the primary subsidiary concept, but in any modelling episode there are others which emerge, as the example below demonstrates.

Modelling - An Illustrative Example

We can illustrate Margot's approach to teaching via modelling, and the subsidiary concepts supporting it, by detailed reference to one example from her interview. She describes at some length one way to approach the teaching of rates of change with sixth form students (age 17 years), in this way:

> What we then try to do again is to make the work as practical as possible, and last year what we did was we took the coil of rope, we took a coil of rope and we, I went out to [store name] and brought all this bits of string and they mark off. So this is bits of rope being round onto a coil and they mark off with pen, and they get a table for the number of the coil and the length of the string.
>
> So what they're actually doing, is they're modelling rope being wound by machine or onto a spool or whatever. Worked beautifully, it was perfect, and then we gave them questions that, we asked them to graph it, so they were graphing, and then we asked them to estimate the rate at which the rope was going on between two integer values, so that they could work from the table.
>
> We then asked them to work out the rate at which it was going on two interpolated values so they actually had to work from that curve, and then we asked them to work out the instantaneous rate at which it was going on, so that they had to have the idea of a tangent. But again, this is all functions because again we're looking at one variable resulting in a change and another variable and the resulting graph and how you interpret it, and when we modelled that on the graphics calculator, it was just beautiful.

The emphasis on modelling is clear here. Not just in Margot's use of the word which indicates that she believes she is encouraging modelling, but in terms of the whole approach. Her aim is to take a real world 'practical' situation and represent it mathematically. In terms of our theoretical model, the focus concept of function is supported by, and related to, subsidiary concepts, each of which has a number of different representations. The underlying subsidiary concepts which she specifically mentions or alludes to for function are: average and instantaneous rate of change; interpolation; chord, tangent; gradient; and variable. The major representations employed are symbolic, tabular and graphical. Margot often sees the value of technology in enabling manipulation of the mathematical concepts both within and between these different representations (Thompson, 1992), but whether the technology is used or not, the transition between representations, preserving the conceptual structure of the mathematics is a crucial one in her schemas. In this example, the variables are first symbolised (one representation) with the function being a relationship between the independent and dependent variables, and then there is a move between representations as the symbols enable access to a tabular representation. Working within this tabular representation the average rate of change is calculated from the two sets of values present. As a next step, Margot uses two values which have not been directly measured to stimulate the use of a graphical representation, with a co-ordinate or ordered pair representation of the data as the link between table and graph. Once in the graphical mode an algorithm to find the gradient of a chord from the use of interpolated values is employed (we note that although this can be done by linear interpolation from the table this was not mentioned here). Finally, also working within the graphical representation, the concept of instantaneous rate of change requiring the graphing of a tangent and an algorithm to find its gradient, was introduced. This example does not involve algebraic symbolisation other than of the variables, since the situation has been adequately represented mathematically without recourse to this (although, of course, one could have gone on to model a polynomial function for the data). The final step in the modelling process involves working within a representation to carry out algorithmic processes, in this case to calculate gradients or rates of change, namely between two points and at a point.

This example, which as stated, is lacking a symbolic, algebraic representation of function is still completely about function for Margot, since as she says "this is all functions because again we're looking at one variable resulting in a change and another variable". This fundamental conceptual mental construct, the linking of an independent and a dependent variable, runs through all her ideas on function.

When we isolate some of the key concepts which Margot is building into the modelling she is doing in the classroom what kind of rich relationships do we see? Figure 2 presents an attempt to represent a macro view of this modelling example and its relationship to our theoretical model. We are currently investigating Margot's schematic thinking by looking at the micro level too, taking some of the focus concepts, such as those in this example, and drawing simplified schemas for them based on Margot's comments in the free recall task. The quality of some of the links in Margot's schemas may be observed in the comments she makes. For example, Margot is definitely not limited to thinking in one representation.

Figure 2
A Macro View of Margot's Modelling Example and its Relationship with the Theoretical Model

Concepts	Physical Examples	Representations	Algorithms
Function	Wind string on spool	Symbolise variables	Calculate gradient of a chord
Variable	Mark length	Construct table of values	Calculate gradient of a tangent
Rate of change (average and instant)	Count number of coils	Convert table to ordered pairs	
Chord, Tangent		Plot Graph	
Gradient			

When she talks she moves seamlessly between them. One significant episode showing the linking between concepts and representations is the way that Margot connects the symbolic form of variable and function, with parameters and the inverse function.

> Well, where you have two variables where x and y are related to one another through a third variable often denoted by the letter t. So for example if you were to graph $x=t$, $y=t^2$ you would actually be graphing $y=x^2$. To get your inverse, all you then have to do is make $x=t^2$, $y=t$, and you get that mirror image. So that works really nicely.

Notice here also her use of the terms "graphing" and "mirror image" referring to a graphical representation, in the context of a manipulation within the symbolic representation whereby she combines $x=t$ and $y=t^2$ to get $y=x^2$. This demonstrates that her schemas are flexible enough to allow her to think and work between representations and that she has schematic links between these and variable, function, parametric form, and inverse function.

DISCUSSION

In this paper we have introduced a model for studying teachers' conceptual knowledge in a given domain in relation to mathematical modelling. In this model we have considered the role of schematic knowledge and how it is important to have high quality, strong connections between the concept under focus and both subsidiary concepts and their representations. It is our contention that it is the richness and robustness of the structure of the teachers' schemas which are a primary influence on whether their teaching of the focus concept is procedural or conceptual. The data presented here suggest that experienced teachers may have more links to subsidiary concepts, and place greater emphasis on these and we hypothesise that these links enable them to move between representations more easily, keeping the focus concept intact, while gaining the advantages that each has to offer. In contrast, an emphasis on procedural aspects of the focus concept would make it more difficult to move across representations and hence tend to anchor one in one representation, for example, the symbolic one. This difference may be illustrated with regard to function by considering two approaches to teaching a graphical solution to the problem: "Find graphically where $y=x^2-3$ is zero"

A teacher, such as Margot, who has a rich schema for function with many subsidiary concepts, will keep alive the concept of a one–to–one (or one–to–many) relationship between variables as the representations are traversed. The table of values, the set of co–ordinates and the graph are each simply viewed as another representation of the symbolically presented functional relationship which assigns a value x^2-3 to a value x, in a one–to–one manner. The question (which could be answered in any representation) then becomes 'Which value of x produces the value 0?' In contrast, a teacher who has less rich schemas may concentrate on a procedural approach which lacks this underlying linkage. A sequence of conceptually unconnected procedures, in or between representations, is the result. Thus, students may calculate values of y given certain values of x. They may transfer these to a graph by a matching procedure which aligns the first number to the x–axis and the second to the y–axis, giving a sequence of points, and then join these up. The final procedure, in the graphical mode, sees them read the value where the curve crosses the x–axis. While the solutions may be the same, the conceptual knowledge built is quite different. The results of our study indicate a potential relationship between teachers' schema and the quality of schemas constructed by students. Chinnappan (1998) showed that, even when students had built up a reasonable number of schemas in the domain of geometry, the quality of their problem search was not flexible enough to construct alternative representations of the given problem. It would seem that these students' schemas had more procedural than conceptual information. We suggest that teachers need to draw more on conceptually-dominated schemas of the type revealed by our experienced teacher in order to promote a more flexible approach to mathematical learning and problem solving by students.

REFERENCES

Anderson, J. R. (1995). *Cognitive psychology and its implications*. Fourth edition. New York: W. H. Freeman & Company.

Ball, D.L. & McDiarmid, G.W. (1990). The subject-matter preparation of teachers. In Houston, W.R. (Ed.) *Handbook of Research on Teacher Education*. New York: McMillan.

Chinnappan, M. (1998). The accessing of geometry schemas by high school students. *Mathematics Education Research Journal, 10*(2), 27-45.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme, *Journal of Mathematical Behavior*, 15, 167-192.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking, In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95–123). Dordrecht: Kluwer Academic Publishers.

Fischbein, E. (1994). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity, In R. Biehler, R.W. Scholz, R. Sträßer, B. Winkelmann (Eds.) *Didactics of Mathematics as a Scientific Discipline*, (pp. 231–245). Dordrecht: Kluwer.

Frederiksen, N. (1984). Implications of cognitive theory for instruction in problem solving. *Review of Educational Research*, *54*, 363-407.

Gray, E. M. and Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic, *The Journal for Research in Mathematics Education, 26(2)*, 115–141.

Hong, Y. Y. & Thomas, M. O. J. (1998). Versatile understanding in integration, Proceedings of the International Congress of Mathematics Instruction – South East Asian Conference on Mathematics Education, Seoul, Korea, 255–265.

Kaput, J. (1992). Technology and Mathematics Education, NCTM Yearbook on Maths. Education, 515-556.

Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for Research in Mathematics Education, 20,* 2-75.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va:The Council

Paas, F. G. W. C. (1992). The training strategies for attaining transfer of problem-solving skill in statistics: A cognitive approach, *Journal of Educational Psychology,* 84(4), 429-434.

Skemp, R. R. (1979). *Intelligence, Learning and Action: - A Foundation For Theory And Practice In Education,* Chichester, UK: Wiley.

Skemp, R. R. (1985). PMP: A progress report, *Proceedings of the 9th Conference of the International Group for the Psychology of Mathematics Education,* Utrecht Netherlands, 447-452.

Sweller, J. (1992). Cognitive theories and their implications to mathematics instruction. In G. Leder (Ed)., *Assessment and Learning of Mathematics,* Victoria: The Australian Council of Educational Research.

Tall, D. O. and Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer, *Educational Studies in Mathematics,* 22, 125-147.

Thomas, M.O.J. (1994). A process-oriented preference in the writing of algebraic equations, *Proceedings of the 17th Mathematics Education Research Group of Australasia Conference,* Lismore, Australia, 599-606.

Endnote

[1] A pseudonym